12

Flow of an ideal fluid

When the Reynolds number Re is large, since the diffusion of vorticity is now small (eqn (6.18)) because the boundary layer is very thin, the overwhelming majority of the flow is the main flow. Consequently, although the fluid itself is viscous, it can be treated as an ideal fluid subject to Euler's equation of motion, so disregarding the viscous term. In other words, the applicability of ideal flow is large.

For an irrotational flow, the velocity potential ϕ can be defined so this flow is called the potential flow. Originally the definition of potential flow did not distinguish between viscous and non-viscous flows. However, now, as studied below, potential flow refers to an ideal fluid.

In the case of two-dimensional flow, a stream function ψ can be defined from the continuity equation, establishing a relationship where the Cauchy–Riemann equation is satisfied by both ϕ and ψ. This fact allows theoretical analysis through application of the theory of complex variables so that ϕ and ψ can be obtained. Once ϕ or ψ is obtained, velocities u and v in the x and y directions respectively can be obtained, and the nature of the flow is revealed.

In the case of three-dimensional flow, the theory of complex variables cannot be used. Rather, Laplace's equation $\Delta^2\phi = 0$ for a velocity potential $\phi = 0$ is solved. Using this approach the flow around a sphere etc. can be determined.

Here, however, only two-dimensional flows will be considered.

12.1 Euler's equation of motion

Consider the force acting on the small element of fluid in Fig. 12.1. Since the fluid is an ideal fluid, no force due to viscosity acts. Therefore, by Newton's second law of motion, the sum of all forces acting on the element in any direction must balance the inertia force in the same direction. The pressure acting on the small element of fluid $dx\,dy$ is, as shown in Fig. 12.1, similar to Fig. 6.3(b). In addition, taking account of the body force and also assuming that the sum of these two forces is equal to the inertial force, the equation of motion for this case can be obtained. This is the case where the

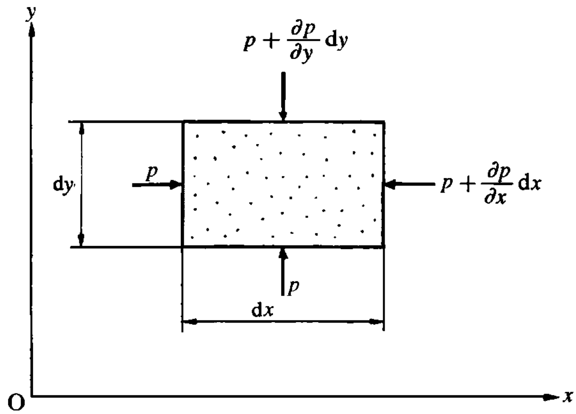

Fig. 12.1 Balance of pressures on fluid element

viscous term of eqn (6.12) is omitted. Consequently the following equations are obtainable:

$$\left.\begin{aligned}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)&=\rho X-\frac{\partial p}{\partial x}\\ \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)&=\rho Y-\frac{\partial p}{\partial y}\end{aligned}\right\}\tag{12.1}$$

These are similar equations to eqn (5.4), and are called Euler's equations of motion for two-dimensional flow.

For a steady flow, if the body force term is neglected, then:

$$\left.\begin{aligned}\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)&=-\frac{\partial p}{\partial x}\\ \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)&=-\frac{\partial p}{\partial y}\end{aligned}\right\}\tag{12.2}$$

If u and v are known, the pressure is obtainable from eqn (12.1) or eqn (12.2).

Generally speaking, in order to obtain the flow of an ideal fluid, the continuity equation (6.2) and the equation of motion (12.1) or eqn (12.2) must be solved under the given initial conditions and boundary conditions. In the flow fluid, three quantities are to be obtained, namely u, v and p, as functions of t and x, y. However, since the acceleration term, i.e. inertial term, is non-linear, it is so difficult to obtain them analytically that a solution can only be obtained for a particular restricted case.

12.2 Velocity potential

The velocity potential ϕ as a function of x and y will be studied. Assume that

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \tag{12.3}$$[1]

From $\partial u/\partial y = \partial^2\phi/\partial y \partial x = \partial^2\phi/\partial x \partial y = \partial v/\partial x$ the following relationship is obtained:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{12.4}$$

This is the condition for irrotational motion. Conversely, if a flow is irrotational, function ϕ as in the following equation must exist for u and v:

$$\mathrm{d}\phi = u\,\mathrm{d}x + v\,\mathrm{d}y \tag{12.5}$$

Using eqn (12.3),

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y \tag{12.6}$$

Consequently, when the function ϕ has been obtained, velocities u and v can also be obtained by differentiation, and thus the flow pattern is found. This function ϕ is called velocity potential, and such a flow is called potential or irrotational flow. In other words, the velocity potential is a function whose gradient is equal to the velocity vector.

Equation (12.6) turns out as follows if expressed in polar coordinates:

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{\partial \phi}{r\,\partial \theta} \tag{12.7}$$

For the potential flow of an incompressible fluid, substitute eqn (12.3) into continuity equations (6.2), and the following relationship is obtained:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{12.8}$$[2]

Equation (12.8), called Laplace's equation, is thus satisfied by the velocity potential used in this manner to express the continuity equation. From any solution which satisfies Laplace's equation and the particular boundary conditions, the velocity distribution can be determined. It is particularly

[1] In general, whenever u, v and w are respectively expressed as $\partial\phi/\partial x$, $\partial\phi/\partial y$ and $\partial\phi/\partial z$ for vector V(x, y and z components are respectively u, v and w), vector V is written as grad ϕ or $\nabla\phi$:

$$V = \mathrm{grad}\,\phi = \nabla\phi = \left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right]$$

Equation (12.3) is the case of two-dimensional flow where $w = 0$, and can be written as grad ϕ or $\nabla\phi$.

[2] That is

$$\mathrm{div}\,\nabla = \mathrm{div}[u, v, w] = \mathrm{div}(\mathrm{grad}\,\phi) = \mathrm{div}\,\nabla\phi = \mathrm{div}\left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right] = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is called the Laplace operator (Laplacian), abbreviated to Δ. Equation (12.8) is for a two-dimensional flow where $w = 0$, expressed as $\Delta\phi = 0$.

noteworthy that the pattern of potential flow is determined solely by the continuity equation and the momentum equation serves only to determine the pressure.

A line along which ϕ has a constant value is called the equipotential line, and on this line, since $d\phi = 0$ and the inner product of both vectors of velocity and the tangential line is zero, the direction of fluid velocity is at right angles to the equipotential line.

12.3 Stream function

For incompressible flow, from the continuity equation (6.2),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{12.9}$$

This is eqn (12.4) but with u and v respectively replaced by $-v$ and u. Consequently, corresponding to eqn (12.5), it turns out that there exists a function ψ for x and y shown by the following equation:

$$d\psi = -v\,dx + u\,dy \tag{12.10}$$

In general, since

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \tag{12.11}$$

u and v are respectively expressed as follows:

$$-v = \frac{\partial \psi}{\partial x} \qquad u = \frac{\partial \psi}{\partial y} \tag{12.12}$$

Consequently, once function ψ has been obtained, differentiating it by x and y gives velocities v and u, revealing the detail of the fluid motion. ψ is called the stream function.

Expressing the above equation in polar coordinates gives

$$v_r = \frac{\partial \psi}{r\,\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \tag{12.13}$$

In general, for two-dimensional flow, the streamline is as follows, from eqn (4.1):

$$\frac{dx}{u} = \frac{dy}{v}$$

or

$$-v\,dx + u\,dy = 0 \tag{12.14}$$

From eqns (12.12) and (12.14), the corresponding $d\psi = 0$, i.e. $\psi = \text{constant}$, defines a streamline. The product of the tangents of a streamline and an equipotential line at the crossing point of both lines is as follow from eqns (12.3) and (12.12):

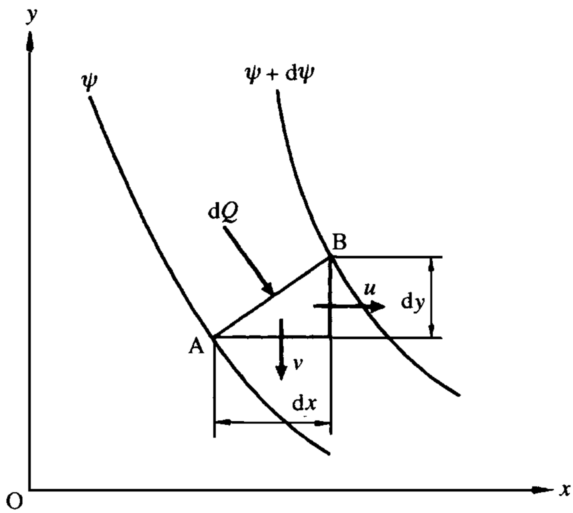

Fig. 12.2 Relationship between flow rate and stream function

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_\phi \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_\psi = \left(\frac{\partial \psi}{\partial x}\Big/\frac{\partial \psi}{\partial y}\right) \times \left(\frac{\partial \phi}{\partial x}\Big/\frac{\partial \phi}{\partial y}\right) = -1$$

This relationship shows that the streamline intersects normal to the equipotential line at the crossing point of the two lines.

As shown in Fig. 12.2, consider points A and B on two closely neighbouring streamlines, ψ and $\psi + \mathrm{d}\psi$. The volume flow rate $\mathrm{d}Q$ flowing in unit time and crossing line AB is as follows from the figure:

$$\mathrm{d}Q = u\,\mathrm{d}y - v\,\mathrm{d}x = \frac{\partial \psi}{\partial y}\mathrm{d}y + \frac{\partial \psi}{\partial x}\mathrm{d}x = \mathrm{d}\psi$$

The volume flow rate Q of fluid flowing between two streamlines $\psi = \psi_1$ and $\psi = \psi_2$ is thus given by the following equation:

$$Q = \int \mathrm{d}Q = \int_{\phi_1}^{\phi_2} \mathrm{d}\psi = \psi_2 - \psi_1 \tag{12.15}$$

Substituting eqn (12.12) into (4.8) for flow without vorticity, the following is obtained, clarifying that the stream function satisfies Laplace's equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{12.16}$$

12.4 Complex potential

For a two-dimensional incompressible potential flow, since the velocity potential ϕ and stream function ψ exist, the following equations result from eqns (12.3) and (12.12):

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{12.17}$$

These equations are called the Cauchy–Riemann equations in the theory of complex variables. In this case they express the relationship between the velocity potential and stream function. The Cauchy–Riemann equations clarify the fact that ϕ and ψ both satisfy Laplace's equation. They also clarify the fact that a combination of ϕ and ψ satisfying the Cauchy–Riemann conditions expresses a two-dimensional incompressible potential flow.

Now, consider a regular function[3] $w(z)$ of complex variable $z = x + \mathrm{i}y$ and express it as follows by dividing it into real and imaginary parts:

$$\left.\begin{aligned} &w(z) = \phi + \mathrm{i}\psi \\ &z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = re^{\mathrm{i}\theta} \\ &\phi = \phi(x, y) \qquad \psi = \psi(x, y) \end{aligned}\right\} \tag{12.18}$$

and ϕ and ψ above satisfy eqn (12.17) owing to the nature of a regular function. Consequently, real part $\phi(x, y)$ and imaginary part $\psi(x, y)$ of the regular function $w(z)$ of complex number z can respectively be regarded as the velocity potential and the stream function of the two-dimensional incompressible potential flow. In other words, there exists an irrotational motion whose equipotential line is $\phi(x, y) = \text{constant}$ and streamline $\psi(x, y) = \text{constant}$. Such a regular function $w(z)$ is called the complex potential.

From eqn (12.18)

$$\begin{aligned} \mathrm{d}w &= \frac{\partial w}{\partial x}\mathrm{d}x + \frac{\partial w}{\partial y}\mathrm{d}y = \left(\frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial \phi}{\partial y} + \mathrm{i}\frac{\partial \psi}{\partial y}\right)\mathrm{d}y \\ &= (u - \mathrm{i}v)\mathrm{d}x + (v + \mathrm{i}u)\mathrm{d}y = (u - \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y) = (u - \mathrm{i}v)\mathrm{d}z \end{aligned}$$

Therefore

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - \mathrm{i}v \tag{12.19}$$

Consequently, whenever $w(z)$ is differentiated with respect to z, as shown in Fig. 12.3, its real part yields velocity u in the x direction, and the negative of its imaginary part yields velocity v in the y direction. The actual velocity $u + \mathrm{i}v$ is called the complex velocity while $u - \mathrm{i}v$ in the above equation is the conjugate complex velocity.

[3] The function whose differential at any point with respect to z is independent of direction in the z plane is called a regular function. A regular function satisfies the Cauchy–Riemann equations.

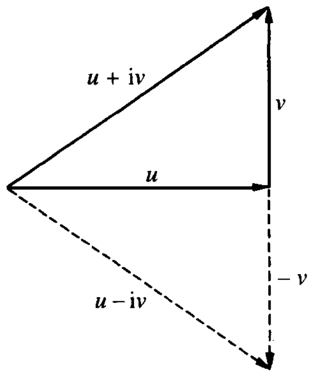

Fig. 12.3 Complex velocity

12.5 Example of potential flow

12.5.1 Basic example

Parallel flow

For the uniform flow U shown in Fig. 12.4, from eqn (12.3)

$$u = \frac{\partial \phi}{\partial x} = U \quad v = \frac{\partial \phi}{\partial y} = 0$$

Therefore

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = U\, dx$$
$$\phi = Ux$$

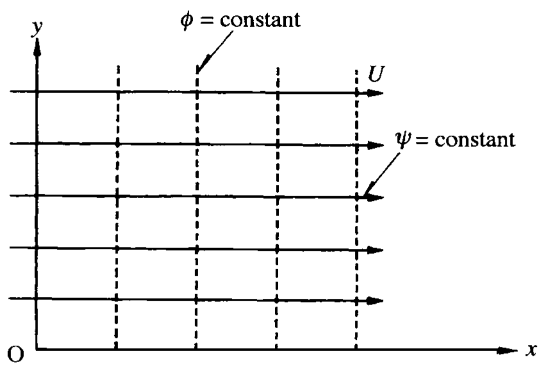

Fig. 12.4 Parallel flow

From eqn (12.12)

$$u = \frac{\partial \psi}{\partial y} = U \quad v = -\frac{\partial \psi}{\partial x} = 0$$

Therefore

$$d\phi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = U\, dy$$
$$\psi = Uy$$
$$w(z) = \phi + i\psi = U(x + iy) = Uz \qquad (12.20)$$

The complex potential of parallel flow U in the x direction emerges as $w(z) = uz$.

Furthermore, if the complex potential is given as $w(z) = Uz$, the conjugate complex velocity is

$$\frac{dw}{dz} = U \qquad (12.21)$$

clarifying again that it expresses a uniform flow in the direction of the x axis.

Source

As shown in Fig. 12.5, consider a case where fluid discharges from the origin (point O) at quantity q per unit time. Putting velocity in the radial direction on a circle of radius r to v_r, the discharge q per unit thickness is

$$q = 2\pi r v_r = \text{constant} \qquad (12.22)$$

From eqns (12.7) and (12.22)

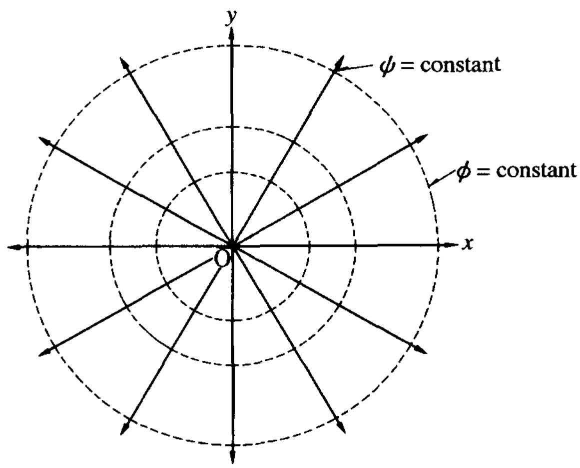

Fig. 12.5 Source

$$v_r = \frac{\partial \phi}{\partial r} = \frac{q}{2\pi r}$$

Also, from eqn (12.7),

$$v_\theta = \frac{\partial \phi}{r\,\partial \theta} = 0$$

Integrating $d\phi$ in the above equation gives

$$\phi = \frac{q}{2\pi} \log r \tag{12.23}$$

Then, from eqns (12.13) and (12.22),

$$v_r = \frac{\partial \psi}{r\,\partial \theta} = \frac{q}{2\pi r} \qquad v_\theta = -\frac{\partial \psi}{\partial r} = 0$$

Therefore

$$\psi = \frac{q}{2\pi}\theta \tag{12.24}$$

Consequently, the complex potential is expressed by the following equation:

$$w = \phi + \mathrm{i}\psi = \frac{q}{2\pi}(\log r + \mathrm{i}\theta) = \frac{q}{2\pi}\log(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{q}{2\pi}\log z \tag{12.25}$$

From eqns (12.23) and (12.24) it is known that the equipotential lines are a set of circles centred at the origin while the streamlines are a set of radial lines radiating from the origin. Also, it is noted that the flow velocity v_r is inversely proportional to the distance r from the origin.

Whenever $q > 0$, fluid flows out evenly from the origin towards the periphery. Such a point is called a source. Conversely, whenever $q < 0$, fluid is absorbed evenly from the periphery. Such a point is called a sink. $|q|$ is called the strength of the source or sink.

Free vortex

In Fig. 12.6, fluid rotates around the origin with tangential velocity v_θ at any given radius r. The circulation Γ is as follows from eqn (4.9):

$$\tilde{A} = \int_{\theta=0}^{2\pi} v_\theta \, ds = v_\theta r \int_0^{2\pi} d\theta = 2\pi r v_\theta$$

The velocity potential ϕ is

$$v_\theta = \frac{\partial \phi}{r\,\partial \theta} = \frac{\tilde{A}}{2\pi r} \qquad v_r = \frac{\partial \phi}{\partial r} = 0$$

Therefore

$$\phi = \frac{\Gamma}{2\pi}\theta \tag{12.26}$$

It emerges that v_θ is inversely proportional to the distance from the centre.

The stream function ψ is

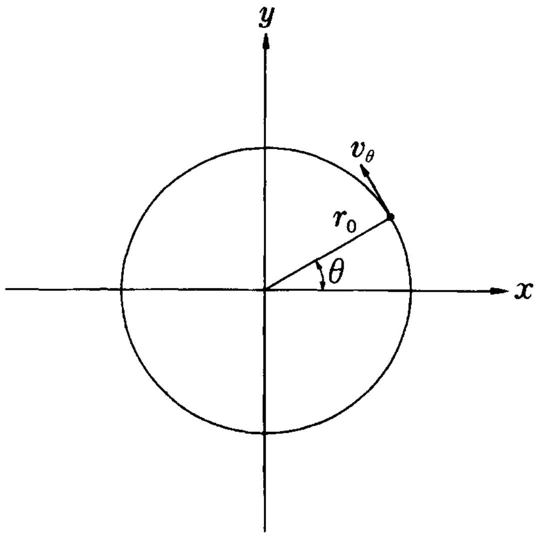

Fig. 12.6 Vortex

$$v_\theta = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r} \quad v_r = \frac{\partial\psi}{r\,\partial\theta} = 0$$

Therefore

$$\psi = -\frac{\Gamma}{2\pi}\log r \tag{12.27}$$

Consequently, the complex potential is

$$w(z) = \phi + \mathrm{i}\psi = \frac{\Gamma}{2\pi}(\theta = \mathrm{i}\log r) = -\frac{\mathrm{i}\Gamma}{2\pi}(\log r + \mathrm{i}\theta) = -\frac{\mathrm{i}\Gamma}{2\pi}\log z \tag{12.28}$$

For clockwise circulation, $w(z) = (\mathrm{i}\Gamma/2\pi)$.

From eqns (12.26) and (12.17), it is known that the equipotential lines are a group of radial straight lines passing through the origin whilst the flow lines are a group of concentric circles centred on the origin. This flow appears in Fig. 12.5 with broken lines representing streamlines and solid lines as equipotential lines. The circulation Γ is positive counterclockwise, and negative clockwise.

This flow consists of rotary motion in concentric circles around the origin with the velocity inversely proportional to the distance from the origin. Such a flow is called a free vortex while the origin point itself is a point vortex. The circulation is also called the strength of the vortex.

12.5.2 Synthesising of flows

When there are two regular functions $w_1(z)$ and $w_2(z)$, the function obtained as their sum

$$w(z) = w_1(z) + w_2(z) \tag{12.29}$$

is also a regular function. If w_1 and w_2 represent the complex potentials of two flows, another complex potential is obtained from their sum. By combining two two-dimensional incompressible potential flows in such a manner, another flow can be obtained.

Combining a source and a sink

Assume that, as shown in Fig. 12.7, the source q is at point A ($z = -a$) and sink $-q$ is at point B ($z = a$).

The complex potential w_1 at any point z due to the source whose strength is q at point A is

$$w_1 = \frac{q}{2\pi}\log(z + a) \tag{12.30}$$

The complex potential w_2 at any point z due to the sink whose strength is q is

$$w_2 = -\frac{q}{2\pi}\log(z - a) \tag{12.31}$$

Because of the linearity of Laplace's equation the complex potential w of the flow which is the combination of these two flows is

$$w = \frac{q}{2\pi}[\log(z + a) - \log(z - a)] \tag{12.32}$$

Now, from Fig. 12.7, since

$$z + a = r_1 e^{i\theta_1} \qquad z - a = r_2 e^{i\theta_2}$$

from eqn (12.32)

$$w = \frac{q}{2\pi}\left(\log\frac{r_1}{r_2} + i(\theta_1 - \theta_2)\right) \tag{12.33}$$

Therefore

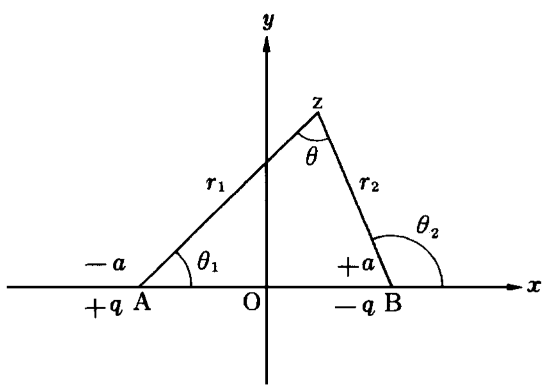

Fig. 12.7 Definition of variables for source A and sink B combination

$$\phi = \frac{q}{2\pi} \log\left(\frac{r_1}{r_2}\right) \tag{12.34}$$

$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) \tag{12.35}$$

Assuming $\phi = \text{constant}$ from the first equation, equipotential lines are obtainable which are Appolonius circles for points A and B (a group of circles whose ratios of distances from fixed points A and B are constant). Taking $\psi = \text{constant}$, streamlines are obtainable which are found to be another set of circles whose vertical angles are the constant angle $(\theta_1 - \theta_2)$ for chord AB (Fig. 12.8).

Consider the case where $a \to 0$ in Fig. 12.8, under the condition of $aq = \text{constant}$. Then from eqn (12.32),

$$w = \frac{q}{2\pi} \log\left(\frac{1 + a/z}{1 - a/z}\right) = \frac{q}{\pi}\left[\frac{a}{z} + \frac{1}{3}\left(\frac{a}{z}\right)^3 + \frac{1}{5}\left(\frac{a}{z}\right)^5 + \cdots\right] = \frac{aq}{\pi z} = \frac{m}{z} \tag{12.36}$$

A flow given by the complex potential of eqn (12.36) is called a doublet, while $m = aq/\pi$ is its strength. The concept of a doublet is the extremity of a source and a sink of equal strength approaching infinitesimally close to each other whilst increasing their strength.

From eqn (12.36),

$$w = \frac{m}{x + iy} = m\frac{x - iy}{x^2 + y^2} \tag{12.37}$$

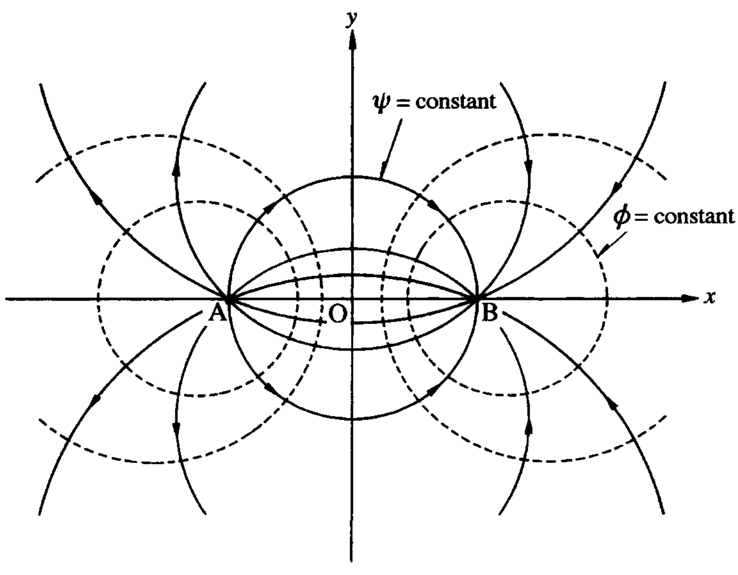

Fig. 12.8 Flow due to the combination of source and sink

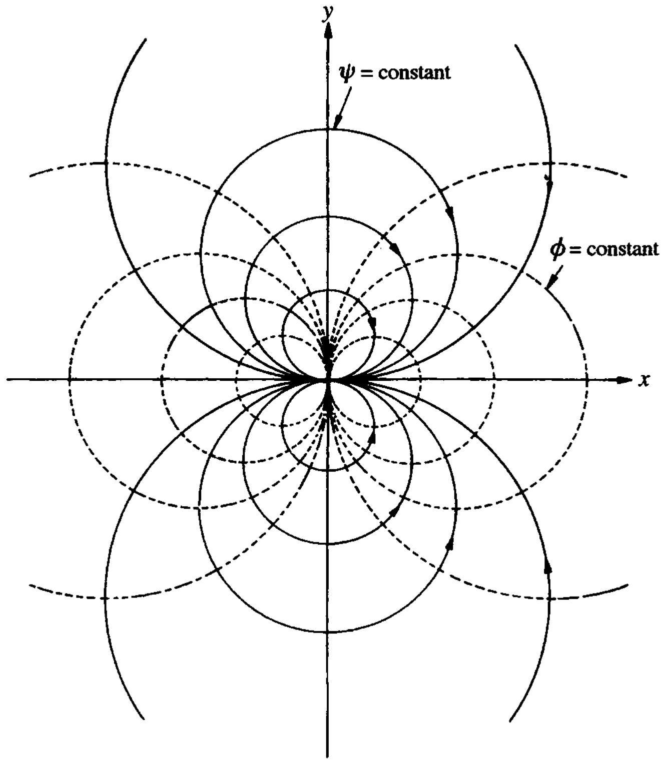

Fig. 12.9 Doublet

$$\phi = \frac{mx}{x^2 + y^2} \tag{12.38}$$

$$\psi = \frac{my}{x^2 + y^2} \tag{12.39}$$

From these equations, as shown in Fig. 12.9, an equipotential line is a circle whose centre is on the x axis whilst being tangential to the y axis, and a streamline is a circle whose centre is on the y axis whilst being tangential to the x axis.

Flow around a cylinder

Consider a circle of radius r_0 centred at the origin in uniform parallel flows. In general, by placing a number of sources and sinks in parallel flows, flows around variously shaped bodies are obtainable. In this case, however, by superimposing parallel flows onto the same doublet shown in Fig. 12.9, flows around a circle are obtainable as follows.

From eqns (12.29) and (12.36) the complex potential when a doublet is in uniform flows U is

$$w(z) = Uz + \frac{m}{z} = U\left(z + \frac{m}{U}\frac{1}{z}\right)$$

Now, put $m/U = r_0^2$, and

$$w(z) = U\left(z + \frac{r_0^2}{z}\right) \tag{12.40}$$

Decompose the above using the relationship $z = r(\cos\theta + \mathrm{i}\sin\theta)$, and

$$w(z) = U\left(r + \frac{r_0^2}{r}\right)\cos\theta + \mathrm{i}U\left(r - \frac{r_0^2}{r}\right)\sin\theta$$

$$\phi = U\left(r + \frac{r_0^2}{r}\right)\cos\theta \tag{12.41}$$

$$\psi = U\left(r - \frac{r_0^2}{r}\right)\sin\theta \tag{12.42}$$

Also, the conjugate complex velocity is

$$\frac{\mathrm{d}w}{\mathrm{d}z} = U - \frac{Ur_0^2}{z^2} \tag{12.43}$$

with stagnation points at $z = \pm r_0$. The streamline passing the stagnation point $\psi = 0$ is given by the following equation:

$$\left(r - \frac{r_0^2}{r}\right)\sin\theta = 0$$

This streamline consists of the real axis and the circle of radius r_0 centred at the origin. By replacing this streamline with a solid surface, the flow around a cylinder is obtained as shown in Fig. 12.10.

The tangential velocity of flow around a cylinder is, from eqn (12.41),

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{r_0^2}{r^2}\right)\sin\theta \tag{12.44}$$

Since $r = r_0$ on the cylinder surface,

$$v_\theta = -2U\sin\theta$$

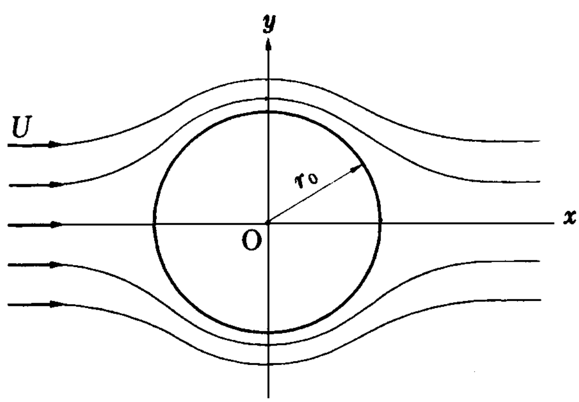

Fig. 12.10 Flow around a cylinder

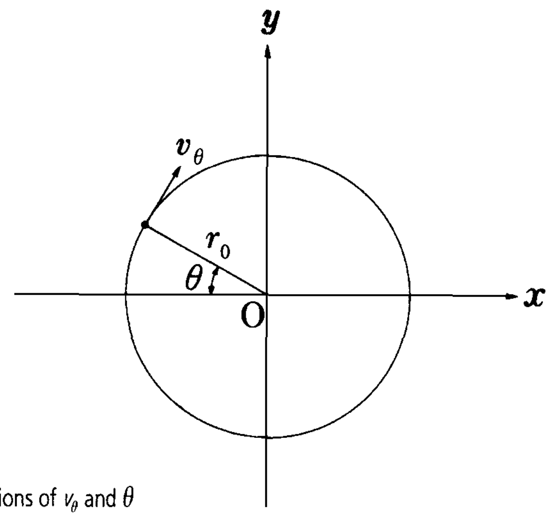

Fig. 12.11 Definitions of v_θ and θ

When the directions of θ and v_θ are arranged as shown in Fig. 12.11, this becomes

$$v_\theta = 2U \sin \theta \tag{12.45}$$

The complex potential when there is clockwise circulation Γ around the cylinder is, as follows from eqns (12.28) and (12.40),

$$w(z) = U\left(z + \frac{r_0^2}{z}\right) + \frac{\mathrm{i}\Gamma}{2\pi} \log z \tag{12.46}$$

The flow in this case turns out as shown in Fig. 12.12. The tangential velocity v'_θ on the cylinder surface is as follows:

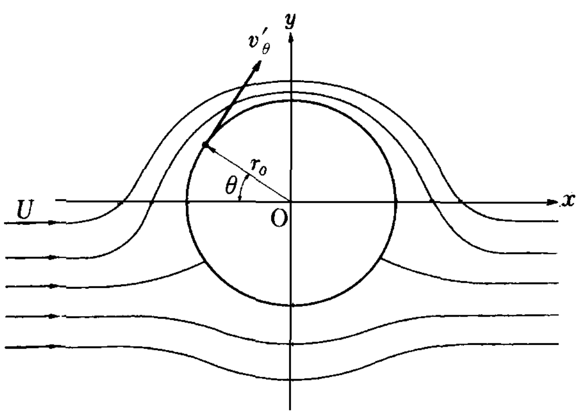

Fig. 12.12 Flow around a cylinder with circulation

$$v'_\theta = 2U \sin\theta + \frac{\Gamma}{2\pi r_0} \tag{12.47}$$

12.6 Conformal mapping

A simple flow can be studied within the limitations of the z plane as in the preceding section. For a complex flow, however, there may be some established cases of useful mapping of a transformation to another plane. For example, by transforming flow around a cylinder etc. through mapping functions onto some other planes, such complex flows as the flow around a wing, and between the blades of a pump, blower or turbine, can be determined.

Assume that there is the relationship

$$\xi = f(z) \tag{12.48}$$

between two complex variables $z = x + \mathrm{i}y$ and $\zeta = \xi + \mathrm{i}\eta$, and that ζ is the regular function of z. Consider a mesh composed of $x =$ constant and $y =$ constant on the z plane as shown in Fig. 12.13. That mesh transforms to another mesh composed of $\xi =$ constant and $\eta =$ constant on the ζ plane. In other words, the pattern on the z plane is different from the pattern on the ζ plane but they are related to each other.

Further, assume that, as shown in Fig. 12.14, point ζ_0 corresponds to point z_0 and that the points corresponding to points z_1 and z_2 both minutely off z_0 are ζ_1 and ζ_2. Then

$$\begin{aligned} z_1 - z_0 &= r_1 \mathrm{e}^{\mathrm{i}\theta_1} \qquad & z_2 = z_0 &= r_2 \mathrm{e}^{\mathrm{i}\theta_2} \\ \zeta_1 - \zeta_2 &= R_1 \mathrm{e}^{\mathrm{i}\beta_1} \qquad & \zeta_2 - \zeta_0 &= R\mathrm{e}^{\mathrm{i}\beta_2} \end{aligned}$$

From eqn (12.48),

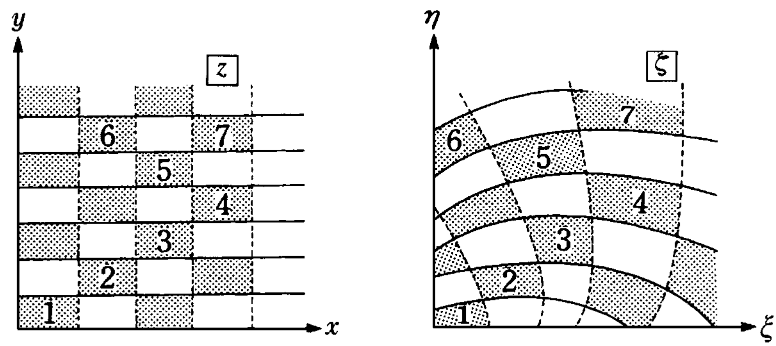

Fig. 12.13 Corresponding mesh on ζ and z planes

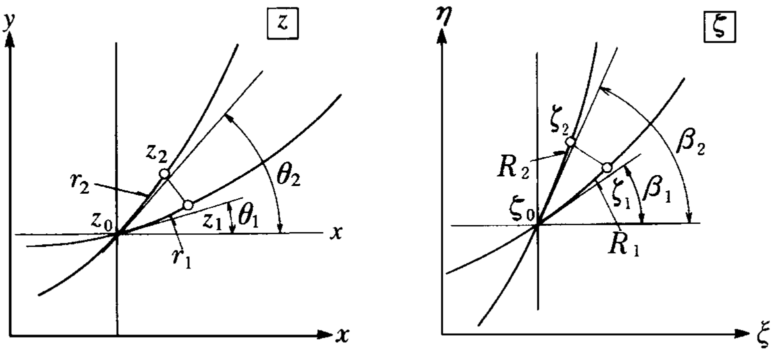

Fig. 12.14 Conformal mapping

$$\lim_{z_1 \to z_2} \left(\frac{\zeta_1 - \zeta_0}{z_1 - z_0} \right) = \left(\frac{d\zeta}{dz} \right)_{z=z_0} = \lim_{z_1 \to z_2} \left(\frac{\zeta_2 - \zeta_0}{z_2 - z_0} \right)$$

or

$$\frac{R_1 e^{i\beta_1}}{r_1 e^{i\theta_1}} = \frac{R_2 e^{i\beta_2}}{r_2 e^{i\theta_2}}$$

From the above, it turns out that

$$\frac{r_2}{r_1} = \frac{R_2}{R_1} \qquad \theta_2 - \theta_1 = \beta_2 - \beta_1$$

and the minute triangles on the z plane are

$$\Delta z_0 z_1 z_2 \propto \Delta \zeta_0 \zeta_1 \zeta_2 \tag{12.49}$$

This shows that even though the pattern as a whole on the z plane may be very different from that on the ζ plane, their minute sections are similar and equiangularly mapped. Such a manner of pattern mapping is called conformal mapping, and $f(z)$ is the mapping function.

Now, consider the mapping function

$$\zeta = z + \frac{a^2}{z} \quad (a > 0) \tag{12.50}$$

Substitute a circle of radius a on the z plane, $z = ae^{i\theta}$, into eqn (12.50),

$$\zeta = a(e^{i\theta} + 1/e^{i\theta}) = a(e^{i\theta} + e^{-i\theta}) = 2a\cos\theta \tag{12.51}$$

At the time when θ changes from 0 to 2π, ζ corresponds in $2a \to 0 \to -2a \to 0 \to 2a$. In other words, as shown in Fig. 12.15(a), the cylinder on the z plane is conformally mapped onto the flat board on the ζ plane. The mapping function in eqn (12.50) is renowned, and is called Joukowski's transformation.

If conformal mapping is made onto the ζ plane using Joukowski's mapping function (12.50) while changing the position and size of a cylinder on the z plane, the shape on the ζ plane changes variously as shown in Fig. 12.15.

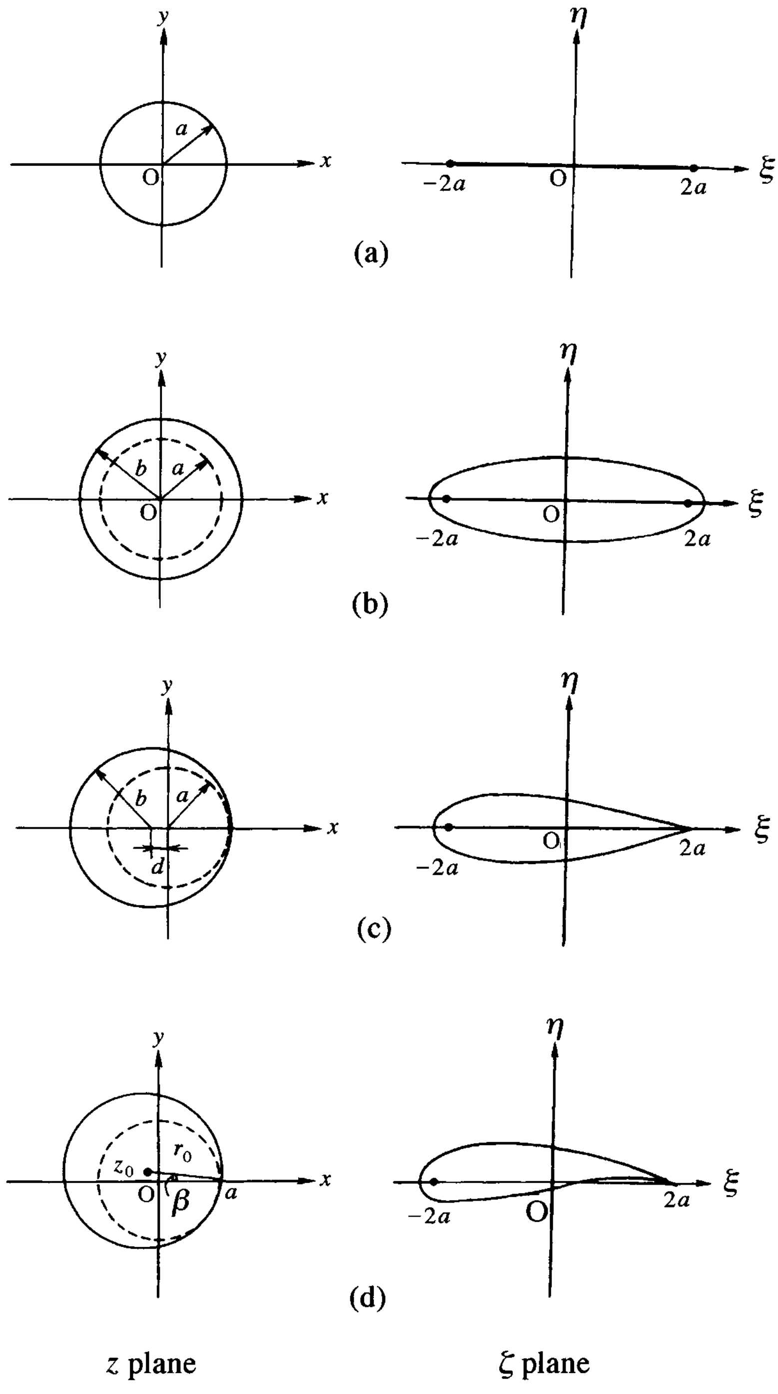

Fig. 12.15 Mapping of cylinders through Joukowski's transformation: (a) flat plate; (b) elliptical section; (c) symmetrical wing; (d) asymmetrical wing

The flow around the asymmetrical wing appearing in Fig. 12.15(d) can be obtained by utilising Joukowski's conversion. Consider the flow in the case where a cylinder of eccentricity z_0 and radius r_0 is placed in a uniform flow U whose circulation strength is Γ. The complex potential of this flow can be obtained by substituting $z - z_0$ for z in eqn (12.46),

$$w = U\left((z - z_0) + \frac{r_0^2}{z - z_0}\right) + \mathrm{i}\frac{\Gamma}{2\pi}\log(z - z_0) \tag{12.52}$$

Putting $z = z_0 + re^{\mathrm{i}\theta}$, from $w = \phi + \mathrm{i}\psi$

$$\phi = U\left(r + \frac{r_0^2}{r}\right)\cos\theta - \frac{\Gamma}{2\pi}\theta \tag{12.53}$$

$$\psi = U\left(r - \frac{r_0^2}{r}\right)\sin\theta - \frac{\Gamma}{2\pi}\log r \tag{12.54}$$

On the circle $r = r_0$, ψ = constant, comprising a streamline. According to the Kutta condition[4] (where the trailing edge must become a stagnation point),

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)_{\substack{\theta=-\beta \\ r=r_0}} = 2Ur_0\sin\beta - \frac{\Gamma}{2\pi} = 0 \tag{12.55}$$

Therefore

$$\Gamma = 4\pi U r_0 \sin\beta \tag{12.56}$$

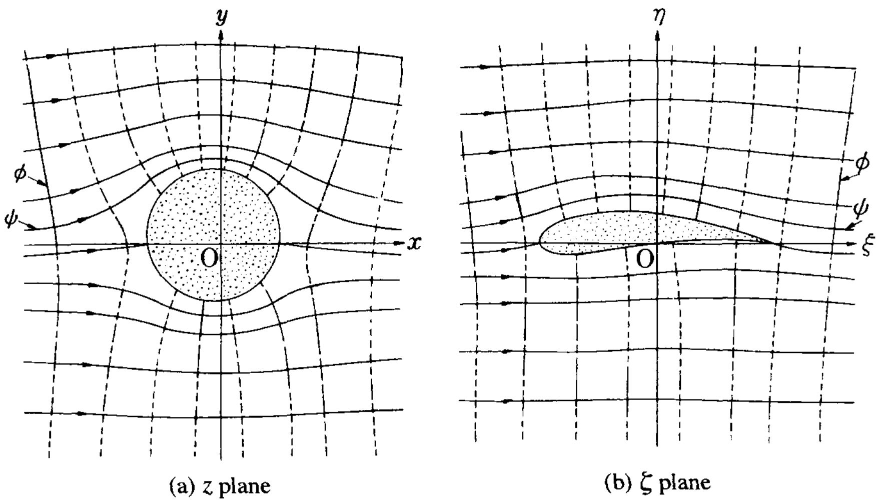

Fig. 12.16 Mapping of flow around cylinder onto flow around wing

[4] If the trailing edge was not a stagnation point, the flow would go around the sharp edge at infinite velocity from the lower face of the wing towards the upper face. The Kutta condition avoids this physical impossibility.

Equipotential lines and streamlines produced by substituting values of Γ satisfying eqn (12.56) into eqns (12.53) and (12.54) are shown in Fig. 12.16(a). They can be conformally mapped onto the ζ plane by utilising Joukowski's conversion by eliminating z from eqns (12.50) and (12.52) to obtain the complex potential on the ζ plane. The resulting flow pattern around a wing can be found as shown in Fig. 12.16(b). In this way, by means of conformal mapping of simple flows, such as around a cylinder, flow around complex-shaped bodies can be found.

Since the existence of analytical functions which shift z to the outside territory of given wing shapes is generally known, the behaviour of flow around these wings can be found from the flow around a cylinder through a process similar to the previous one. In addition, there are examples where it can be used for computing the contraction coefficient[5] of flow out of an orifice in a large vessel and the drag[6] due to the flow behind a flat plate normal to the flow.

12.7 Problems

1. Obtain the velocity potential and the flow function for a flow whose components of velocity in the x and y directions at a given point in the flow are u_0 and v_0 respectively.
2. Show the existence of the following relationship between flow function ψ and the velocity components v_r, v_θ in a two-dimensional flow:

$$v_\theta = -\frac{\partial\psi}{\partial r} \qquad v_r = \frac{\partial\psi}{r\,\partial\theta}$$

3. What is the flow whose velocity potential is expressed as $\phi = \Gamma\theta/2\pi$?
4. Obtain the velocity potential and the stream function for radial flow from the origin at quantity q per unit time.
5. Assuming that $\psi = U(r - r_0^2/r)\sin\theta$ expresses the stream function around a cylinder of radius r_0 in a uniform flow of velocity U, obtain the velocity distribution and the pressure distribution on the cylinder surface.
6. Obtain the pattern of flow whose complex potential is expressed as $w = x^2$.
7. What is the flow expressed by the following complex potential?

$$w(z) = \phi + \mathrm{i}\psi = \frac{\mathrm{i}\Gamma}{2\pi}\log z$$

[5] Lamb, H., *Hydrodynamics*, (1932), 6th edition, 98, Cambridge University Press.
[6] Kirchhoff, G., *Grelles Journal*, 70 (1869), 289.

8. Obtain the complex potential of a uniform flow at angle α to the x axis.

9. Obtain the streamline $y = k$ and the equipotential line $x = c$ of a flow parallel to the x axis on the z plane when mapped onto the ζ plane by mapping function $\zeta = 1/z$.

10. Obtain the flow in the case where parallel flow $w = Uz$ on the z plane is mapped onto the ζ plane by mapping function $\zeta = z^{1/3}$.